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Evaluation of forces in magnetic materials by means of energy and co-energy methods

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Abstract. The evaluation of the total force of magnetic origin acting upon a body in a stationary magnetic field is often carried out using the so-called *magnetic energy* (or *co-energy*) method, which is based on the derivation of the magnetic energy (or co-energy) with respect to a virtual rigid displacement of the considered body. The application of this method is usually justified by resorting to the energy conservation principle, written in terms both of electrical and of mechanical quantities. In this paper we shall re-examine the whole matter in the context of classical thermodynamics, in order to obtain a more comprehensive and general proof of the validity of the energy (or co-energy) approach and to point out its limitations. Two typical configurations will be discussed; in the first one, the field sources are represented by conducting bodies carrying free currents, whereas in the second one a permanent magnet creates the driving field. All magnetic materials are assumed to be non-hysteretic and permanent magnets are represented by means of the well-known linear model in the second quadrant of the (B,H) plane.

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1 Introduction

One of the most popular methods for computing the resultant force acting upon a ferromagnetic body subjected to a magnetostatic field is the so-called *magnetic energy* method (MEM) [1–3]. According to this method, the force is derived from the variation of the magnetic energy (or, alternatively, co-energy) caused by a suitable infinitesimal rigid displacement of the considered body. The use of the energy rather than the co- energy is determined by the nature of the transformation performed: if the magnetic fluxes are kept constant during the displacement of the body, the magnetic energy must be employed, whereas the magnetic co-energy should be used if, on the contrary, the magnetic fluxes can vary and the currents remain unchanged.

The energy method, called frequently *virtual work method*, is widely adopted also in computational electromagnetism, as the most diffused commercial electromagnetic codes for computer aided design of electromagnetic devices are provided with tools or subroutines for energy calculation and the consequent application of MEM at least as an alternative to the surface integration of the Maxwell's stress tensor (see *e.g.* [4]).

However, in spite of this large popularity and diffusion, it is difficult to find in electromagnetic literature any well-established and rigorous proof of the validity of the method. The typical justification that is generally proposed by classical textbooks on electrodynamics (see *e.g.* [1,3]) is based on the energy conservation principle over a magnetic system in which mechanical deformations, exchanges of heat and changes of temperature as well as of thermodynamic internal energy are neglected during the virtual displacement. But these simplifications are, in reality, untenable, as shown by Bobbio in [5] dealing with non-linear, anisotropic, dielectric solids, since most of the physical quantities involved in the energy balance are mixed thermodynamic-magnetic terms.

The aim of this paper is to settle the argument in the more general framework of a comprehensive theory including not only electromagnetism, but also continuum mechanics and thermodynamics. In this context, a detailed proof of the correctness of MEM will be derived and its limitations will be highlighted.

We shall examine two classical operating configurations of electromechanical systems: in the first one several ferromagnetic bodies are placed in the free space and suitable current sources create the driving field, whereas in the second one the ferromagnetic bodies are subjected to the field produced by a permanent magnet. A special attention will be devoted to this latter situation, since the question of the correct definition of the energy stored in a permanent magnet has been recently the subject of a scientific dispute [6,7]. In particular, it will be shown that MEM can be applied if a linear $\mathbf{B} = \mathbf{B}(\mathbf{H})$ characteristics is chosen to describe the physical behavior of the magnet.

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The paper is organized as follows. In Section 2 the concepts of magnetostatic field energy and of thermodynamic internal energy of matter are briefly recalled, since they play a fundamental role in the developed theory. Then, in Sections 3 and 4, the justification of the energy method is provided for the two examined and aforementioned configurations. Finally, in Section 5 some concluding remarks are drawn.

2 The magnetostatic field energy and the thermodynamic internal energy of matter

By definition, the energy stored in the field of a given magnetostatic system corresponds to the work to be spent by a fictitious external operator on assembling the given field sources in an arbitrary way, by counterbalancing only the *long-range* magnetostatic interactions [8,9]. As far as the starting point of the process is concerned, it is assumed that all the field sources are initially diluted at infinity with an infinitesimal volume (or surface) density. *Short-range* interactions, which occur in presence of matter and depend on the microscopic structure of the matter itself, are not included in such a definition since they are taken into account by the thermodynamic internal energy of the considered physical system. Mentioning Bobbio in his book [9], we can say that "the system is therefore conceived as composed of two protagonists that mutually interact: namely, *matter* and *field*. Each of them has its specific content of energy: that of matter is the *thermodynamic internal energy*, whereas that of the field is the *magnetostatic field energy*".

According to such definition and under the assumption to represent magnetized matter as a continuous distribution of Coulombian magnetic dipoles, it is possible to derive the following expression for the differential of the specific magnetostatic field energy $\mathrm{d}w_\mathrm{f}$ [9]:

$$\mathrm{d}w_\mathrm{f} = \mu_0 \mathbf{H}\cdot \mathrm{d}\mathbf{H}. \tag{1}$$

As far as the energy content of matter is concerned, the differential of the specific thermodynamic internal energy can be expressed for a rigid body as a function of the magnetization density $\mathbf{M}$ and of the entropy per unit volume s, considered both as state variables:

$$\mathrm{d}u(s,\mathbf{M}) = \mathbf{H}\cdot \mathrm{d}\mathbf{M} + T\mathrm{d}s, \tag{2}$$

where T is the absolute temperature of the body.

This expression, assumed by Landau and Lifshitz [8], who wrote the differential of the total energy per unit volume u_TOT of the whole system *matter* plus field

$$\mathrm{d}u_\mathrm{TOT}(s,\mathbf{M}) = \mathbf{H}\cdot \mathrm{d}\mathbf{B} + T\mathrm{d}s, \tag{3}$$

and inferred by Bozorth [10], who defined $\mathbf{H}\mathrm{d}\mathbf{M}$ as energy of *magnetization* by means of an energy balance in presence of a magnetization process, has been derived recently in a more general way by Bobbio [9], starting from the expression of force density in matter.

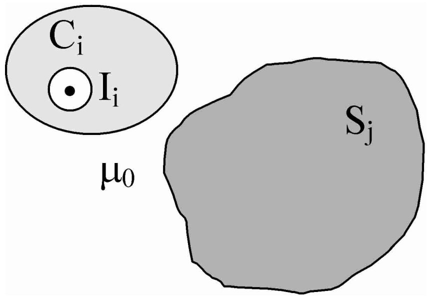

Fig. 1. Geometry of the model problem driven by the field of suitable current sources.

Introducing now the *Helmholtz free energy* per unit volume as:

$$a = u - Ts, \tag{4}$$

and replacing it into (2), it results in

$$\mathrm{d}a(T,\mathbf{M}) = \mathbf{H}\cdot \mathrm{d}\mathbf{M} - s\mathrm{d}T, \tag{5}$$

which represents the differential of the matter free energy.

Adding (1) to (5), it is possible to define the total free energy of the whole system *matter* plus field

$$\mathrm{d}a_\mathrm{TOT}(T,\mathbf{M}) = \mu_0\mathbf{H}\cdot \mathrm{d}\mathbf{H} + \mathbf{H}\cdot \mathrm{d}\mathbf{M} - s\mathrm{d}T. \tag{6}$$

Integrating (6) over a volume Ω, under the assumption of uniform temperature in Ω, the following relationship is derived:

$$\begin{aligned}\mathrm{d}A_\mathrm{TOT} &= \mathrm{d}\iiint_\Omega \left[\int_0^\mathbf{H} \mu_0\mathbf{H}\cdot \mathrm{d}\mathbf{H}\right]\mathrm{d}\Omega \\ &\quad + \mathrm{d}\iiint_\Omega \left[\int_0^\mathbf{M} \mathbf{H}\cdot \mathrm{d}\mathbf{M}\right]\mathrm{d}\Omega - S\mathrm{d}T \\ &= \mathrm{d}\iiint_\Omega \left[\int_0^\mathbf{B} \mathbf{H}\cdot \mathrm{d}\mathbf{B}\right]\mathrm{d}\Omega - S\mathrm{d}T,\end{aligned} \tag{7}$$

where the latter identity has been obtained making use of the equation $\mathbf{M} = \mathbf{B} - \mu_0\mathbf{H}$ defining the magnetization density.

It should be highlighted that $\mathrm{d}A_{\mathrm{TOT}|T=\mathrm{const.}}$ is the usual expression of the so-called magnetic energy, but actually this is a mixed magnetic - thermodynamic term.

3 Magnetic system driven by current sources

Let us consider the magnetic system depicted in Figure 1, where n conducting solids C_1, C_2, ..., C_n carrying respectively the stationary currents I_1, I_2, ..., I_n, and m ferromagnetic bodies S_1, S_2, ..., S_m lie in free space.

Under the assumption that all materials exhibit non-hysteretic properties, the constitutive relationships of the m ferromagnetic bodies can be written in the most general way as

$$\mathbf{B} = \mathbf{B}_j(\mathbf{H}, \bar{\mathbf{E}}, T), \quad \text{for } j = 1, 2, ..., m, \tag{8}$$

or, equivalently as:

$$\mathbf{H} = \mathbf{H}_j(\mathbf{B}, \bar{\mathbf{E}}, T), \quad \text{for } j = 1, 2, ..., m, \tag{9}$$

where $\bar{\mathbf{E}}$ is the strain tensor giving the state of mechanical deformation of the body and T is the absolute temperature of the body itself. Of course, if rigid bodies are considered, the strain tensor is constant.

The state of the system is characterized by:

- the values of the l mechanical degrees of freedom, namely the generalized coordinates q_k, with $k = 1, 2, ..., l$ (see Appendix);
- the n values of currents;
- the $(n+m)$ values of bodies temperature.

For a given set of these state variables, a unique field solution can be found, provided that the Jacobian matrix of the functions $\mathbf{B}_j(\mathbf{H}, \bar{\mathbf{E}}, T)$ with respect to the variable $\mathbf{H}$ (for constant $\bar{\mathbf{E}}$ and T) is symmetric [5].

As far as the forces applied on each body are concerned, they can be of the following kind:

- mechanical active forces external to the whole system;
- magnetic forces among the bodies;
- forces exerted by holonomic constraints, if present (these forces do not make work on the system).

As can be noted, mechanical interactions among the bodies are not taken into account by this model, since we are interested in deriving the expressions for the magnetic forces only.

Let us analyze the *equilibrium condition* of the whole system. This can be accomplished by writing down the equations of statics for each conducting and ferromagnetic body:

$$\mathbf{F}_i^{(\text{ext})} + \mathbf{F}_i^{(\text{mag})} + \Phi_i = 0 \quad \text{for } i = 1, 2, ..., n, \tag{10}$$

$$\mathbf{M}_{\text{G},i}^{(\text{ext})} + \mathbf{M}_{\text{G},i}^{(\text{mag})} + \mathbf{M}_{\text{G},i}^{(\Phi)} = 0 \quad \text{for } i = 1, 2 ..., n, \tag{11}$$

$$\mathbf{F}_j^{(\text{ext})} + \mathbf{F}_j^{(\text{mag})} + \Phi_j = 0 \quad \text{for } j = 1, 2 ..., m, \tag{12}$$

$$\mathbf{M}_{\text{G},j}^{(\text{ext})} + \mathbf{M}_{\text{G},j}^{(\text{mag})} + \mathbf{M}_{\text{G},j}^{(\Phi)} = 0 \quad \text{for } j = 1, 2 ..., m, \tag{13}$$

where $\mathbf{F}_i^{(\text{ext})} \left(\mathbf{M}_{\text{G},i}^{(\text{ext})}\right)$, $\mathbf{F}_i^{(\text{mag})} \left(\mathbf{M}_{\text{G},i}^{(\text{mag})}\right)$, $\Phi_i \left(\mathbf{M}_{\text{G},i}^{(\Phi)}\right)$ are, respectively, the external resultant force, the magnetic resultant force and the resultant force of constraints (resultant moments with respect to the pole G) acting on the ith conductor and $\mathbf{F}_j^{(\text{ext})} \left(\mathbf{M}_{\text{G},j}^{(\text{ext})}\right)$, $\mathbf{F}_j^{(\text{mag})} \left(\mathbf{M}_{\text{G},j}^{(\text{mag})}\right)$, $\Phi_j \left(\mathbf{M}_{\text{G},j}^{(\Phi)}\right)$ are the analogous quantities for the jth ferromagnetic body.

Multiplying the previous relations for a set of virtual displacements $\{\delta\mathbf{G}_i, \delta\mathbf{G}_j\}$ and infinitesimal rotation vectors $\{\varepsilon_i, \varepsilon_j\}$ and adding them up, we obtain the following equation

$$\begin{aligned} &\sum_{i=1}^{n} \left(\mathbf{F}_i^{(\text{ext})} + \mathbf{F}_i^{(\text{mag})}\right) \cdot \delta\mathbf{G}_i + \sum_{j=1}^{m} \left(\mathbf{F}_j^{(\text{ext})} + \mathbf{F}_j^{(\text{mag})}\right) \cdot \delta\mathbf{G}_j \\ &+ \sum_{i=1}^{n} \left(\mathbf{M}_{\text{G},i}^{(\text{ext})} + \mathbf{M}_{\text{G},i}^{(\text{mag})}\right) \cdot \varepsilon_i + \sum_{j=1}^{m} \left(\mathbf{M}_{\text{G},j}^{(\text{ext})} + \mathbf{M}_{\text{G},j}^{(\text{mag})}\right) \varepsilon_j \\ &= 0, \forall \{\delta\mathbf{G}_i, \varepsilon_i, \delta\mathbf{G}_j, \varepsilon_j\} \cdot \end{aligned} \tag{14}$$

But $\{\delta\mathbf{G}_i, \varepsilon_i, \delta\mathbf{G}_j, \varepsilon_j\}$ can be expressed as a function of the generalized coordinates q_k:

$$\delta\mathbf{G}_i = \sum_{k=1}^{l} \frac{\partial \mathbf{G}_i}{\partial q_k} \delta q_k, \tag{15}$$

$$\delta\mathbf{G}_j = \sum_{k=1}^{l} \frac{\partial \mathbf{G}_j}{\partial q_k} \delta q_k, \tag{16}$$

$$\varepsilon_i = \sum_{k=1}^{l} \frac{\partial \varepsilon_i}{\partial q_k} \delta q_k, \tag{17}$$

$$\varepsilon_j = \sum_{k=1}^{l} \frac{\partial \varepsilon_j}{\partial q_k} \delta q_k. \tag{18}$$

The substitution of (15–18) into (14) leads to

$$\begin{aligned} &\sum_{i=1}^{n} \left[\left(\mathbf{F}_i^{(\text{ext})} + \mathbf{F}_i^{(\text{mag})}\right) \cdot \sum_{k=1}^{l} \frac{\partial \mathbf{G}_i}{\partial q_k} \delta q_k\right] \\ &+ \sum_{j=1}^{m} \left[\left(\mathbf{F}_j^{(\text{ext})} + \mathbf{F}_j^{(\text{mag})}\right) \cdot \sum_{k=1}^{l} \frac{\partial \mathbf{G}_j}{\partial q_k} \delta q_k\right] \\ &+ \sum_{i=1}^{n} \left[\left(\mathbf{M}_{\text{G},i}^{(\text{ext})} + \mathbf{M}_{\text{G},i}^{(\text{mag})}\right) \cdot \sum_{k=1}^{l} \frac{\partial \varepsilon_i}{\partial q_k} \delta q_k\right] \\ &+ \sum_{j=1}^{m} \left[\left(\mathbf{M}_{\text{G},j}^{(\text{ext})} + \mathbf{M}_{\text{G},j}^{(\text{mag})}\right) \cdot \sum_{k=1}^{l} \frac{\partial \varepsilon_i}{\partial q_k} \delta q_k\right] = 0. \end{aligned} \tag{19}$$

Inverting the order of summation, (19) can be rewritten as

$$\begin{aligned} &\sum_{k=1}^{l} \left\{ \sum_{i=1}^{n} \left[\mathbf{F}_i^{(\text{ext})} \cdot \frac{\partial \mathbf{G}_i}{\partial q_k} + \mathbf{M}_{\text{G},i}^{(\text{ext})} \cdot \frac{\partial \varepsilon_i}{\partial q_k}\right] \right. \\ &+ \sum_{j=1}^{m} \left[\mathbf{F}_j^{(\text{ext})} \cdot \frac{\partial \mathbf{G}_j}{\partial q_k} + \mathbf{M}_{\text{G},j}^{(\text{ext})} \cdot \frac{\partial \varepsilon_j}{\partial q_k}\right] \\ &+ \sum_{i=1}^{n} \left[\mathbf{F}_i^{(\text{mag})} \cdot \frac{\partial \mathbf{G}_i}{\partial q_k} + \mathbf{M}_{\text{G},i}^{(\text{mag})} \cdot \frac{\partial \varepsilon_i}{\partial q_k}\right] \\ &\left. + \sum_{j=1}^{m} \left[\mathbf{F}_j^{(\text{mag})} \cdot \frac{\partial \mathbf{G}_j}{\partial q_k} + \mathbf{M}_{\text{G},j}^{(\text{mag})} \cdot \frac{\partial \varepsilon_j}{\partial q_k}\right] \right\} \delta q_k = 0. \end{aligned} \tag{20}$$

Defining now $P_k^{(\text{ext})}$ and $P_k^{(\text{mag})}$ as the total components along the q_k degree of freedom respectively of the external and of the magnetic *generalized forces* [11], we have:

$$P_k^{(\mathrm{ext})} = \sum_{i=1}^{n} \left[\mathbf{F}_i^{(\mathrm{ext})} \cdot \frac{\partial \mathbf{G}_i}{\partial q_k} + \mathbf{M}_{\mathrm{G},i}^{(\mathrm{ext})} \cdot \frac{\partial \varepsilon_i}{\partial q_k} \right] + \sum_{j=1}^{m} \left[\mathbf{F}_j^{(\mathrm{ext})} \cdot \frac{\partial \mathbf{G}_j}{\partial q_k} + \mathbf{M}_{\mathrm{G},j}^{(\mathrm{ext})} \cdot \frac{\partial \varepsilon_j}{\partial q_k} \right], \quad (21)$$

$$P_k^{(\mathrm{mag})} = \sum_{i=1}^{n} \left[\mathbf{F}_i^{(\mathrm{mag})} \cdot \frac{\partial \mathbf{G}_i}{\partial q_k} + \mathbf{M}_{\mathrm{G},i}^{(\mathrm{mag})} \cdot \frac{\partial \varepsilon_i}{\partial q_k} \right] + \sum_{j=1}^{m} \left[\mathbf{F}_j^{(\mathrm{mag})} \cdot \frac{\partial \mathbf{G}_j}{\partial q_k} + \mathbf{M}_{\mathrm{G},j}^{(\mathrm{mag})} \cdot \frac{\partial \varepsilon_j}{\partial q_k} \right]. \quad (22)$$

Equation (20) becomes:

$$\sum_{k=1}^{l} \left(P_k^{(\mathrm{ext})} + P_k^{(\mathrm{mag})} \right) \delta q_k = 0. \quad (23)$$

On the other hand, one should be easily convinced that:

$$\delta L_{\mathrm{ext}} = \sum_{k=1}^{l} P_k^{(\mathrm{ext})} \delta q_k, \quad (24)$$

and

$$\delta L_{\mathrm{mag}} = \sum_{k=1}^{l} P_k^{(\mathrm{mag})} \delta q_k. \quad (25)$$

Defining then the elementary work δL_{cs} performed by the current sources during the virtual displacements $\{\delta \mathbf{G}_i, \delta \mathbf{G}_j\}$ or virtual rotations $\{\varepsilon_i, \varepsilon_j\}$ as

$$\delta L_{\mathrm{cs}} = \sum_{i=1}^{n} I_i \mathrm{d}\varphi_i, \quad (26)$$

we can finally write the first principle of thermodynamics

$$\delta L_{\mathrm{ext}} + \delta L_{\mathrm{cs}} + \delta Q = \mathrm{d}W_{\mathrm{f}} + \sum_{i=1}^{n} \mathrm{d}U_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}U_j^{(\mathrm{S})}, \quad (27)$$

where W_{f} is the energy stored in the magnetostatic field and $U_i^{(\mathrm{C})}$ and $U_j^{(\mathrm{S})}$ are the thermodynamic internal energies respectively of the ith conducting body and of the jth ferromagnetic body.

The term δQ can be rewritten according to the second principle of thermodynamics, which states that

$$\delta Q = \sum_{i=1}^{n} T_i^{(\mathrm{C})} \mathrm{d}S_i^{(\mathrm{C})} + \sum_{j=1}^{m} T_j^{(\mathrm{S})} \mathrm{d}S_j^{(\mathrm{S})}, \quad (28)$$

as the transformation between the two static configurations can be thought as reversible ($T_i^{(\mathrm{C})}$ and $T_j^{(\mathrm{S})}$ are the absolute temperatures respectively of the ith conducting body and of the jth ferromagnetic body and $S_i^{(\mathrm{C})}$ and $S_j^{(\mathrm{S})}$ their entropies).

Observing now that (23) dictates that

$$\delta L_{\mathrm{ext}} = -\delta L_{\mathrm{mag}}, \quad (29)$$

and expressing δL_{mag} according to (25), (27) becomes:

$$-\sum_{k=1}^{l} P_k^{(\mathrm{mag})} \delta q_k + \sum_{i=1}^{n} I_i \mathrm{d}\varphi_i = \mathrm{d}W_{\mathrm{f}} + \sum_{i=1}^{n} \mathrm{d}U_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}U_j^{(\mathrm{S})} - \sum_{i=1}^{n} T_i^{(\mathrm{C})} \mathrm{d}S_i^{(\mathrm{C})} - \sum_{j=1}^{m} T_j^{(\mathrm{S})} \mathrm{d}S_j^{(\mathrm{S})}. \quad (30)$$

This expression can be rewritten in terms of the free energies respectively of the n conducting bodies and of the m ferromagnetic bodies taking into account (4):

$$-\sum_{k=1}^{l} P_k^{(\mathrm{mag})} \delta q_k + \sum_{i=1}^{n} I_i \mathrm{d}\varphi_i = \mathrm{d}W_{\mathrm{f}} + \sum_{i=1}^{n} \mathrm{d}A_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\mathrm{S})} + \sum_{i=1}^{n} S_i^{(\mathrm{C})} \mathrm{d}T_i^{(\mathrm{C})} + \sum_{j=1}^{m} S_j^{(\mathrm{S})} \mathrm{d}T_j^{(\mathrm{S})} = \mathrm{d}A_{\mathrm{TOT}} + \sum_{i=1}^{n} S_i^{(\mathrm{C})} \mathrm{d}T_i^{(\mathrm{C})} + \sum_{j=1}^{m} S_j^{(\mathrm{S})} \mathrm{d}T_j^{(\mathrm{S})}, \quad (31)$$

where A_{TOT} is the total Helmholtz free energy defined on the basis of (6) as:

$$\mathrm{d}A_{\mathrm{TOT}} = \mathrm{d}W_{\mathrm{f}} + \sum_{i=1}^{n} \mathrm{d}A_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\mathrm{S})}. \quad (32)$$

If now we introduce a *modified total free energy* A'_{TOT} so that:

$$\mathrm{d}A'_{\mathrm{TOT}} = \mathrm{d}\left(\sum_{i=1}^{n} \phi_i I_i - A_{\mathrm{TOT}} \right), \quad (33)$$

one has:

$$\mathrm{d}A'_{\mathrm{TOT}} = \sum_{k=1}^{l} P_k^{(\mathrm{mag})} \delta q_k + \sum_{i=1}^{n} \varphi_i \mathrm{d}I_i + \sum_{i=1}^{n} S_i^{(\mathrm{C})} \mathrm{d}T_i^{(\mathrm{C})} + \sum_{j=1}^{m} S_j^{(\mathrm{S})} \mathrm{d}T_j^{(\mathrm{S})}, \quad (34)$$

and thus the expression for $P_k^{(\mathrm{mag})}$ can be obtained by means of the following relationship

$$P_k^{(\mathrm{mag})} = \left. \frac{\partial A'_{\mathrm{TOT}}}{\partial q_k} \right|_{\substack{q_r = \mathrm{const.,\ for}\ r = 1..1, r \neq k \\ I_i = \mathrm{const.,\ for}\ i = 1..n \\ T_j = \mathrm{const.,\ for}\ j = 1..m \\ T_i = \mathrm{const.,\ for}\ i = 1..n}} . \quad (35)$$

Let us notice now that the expression for the modified total free energy A'_{TOT} can be simplified under the assumption of constant temperature for each conducting and ferromagnetic body. In such conditions, it happens that:

$$\mathrm{d}A_i^{(\mathrm{C})} = 0, \quad \text{for } i = 1, 2, ..., n, \tag{36}$$

since no variation of the Helmholtz free energy can arise in a rigid conducting body at constant temperature [8,9]. In addition:

$$\mathrm{d}A_j^{(\mathrm{S})} = \iiint\limits_{\Omega_j} \left[\int\limits_0^{\mathbf{M}_j} \mathbf{H}_j(\mathbf{M}, T) \cdot \mathrm{d}\mathbf{M} \right] \mathrm{d}\Omega_j, \text{ for } j{=}1, 2, ..., m, \tag{37}$$

for a process of magnetization taking place at constant temperature in a rigid ferromagnetic body that occupies the volume Ω_j [9].

The expression for the variation of A'_{TOT} becomes therefore:

$$\begin{aligned}\mathrm{d}A'_{\mathrm{TOT}} &= \mathrm{d}\sum_{i=1}^{n} \phi_i I_i - \mathrm{d}W_{\mathrm{f}} \\ &\quad - \sum_{j=1}^{m} \mathrm{d} \iiint\limits_{\Omega_j} \left[\int_0^{\mathbf{M}_j} \mathbf{H}_j(\mathbf{M}, T) \cdot \mathrm{d}\mathbf{M} \right] \mathrm{d}\Omega_j \\ &= \mathrm{d}\sum_{i=1}^{n} \phi_i I_i - \mathrm{d} \left[\iiint\limits_{\Omega_j} \frac{1}{2}\mu_0 H^2 \mathrm{d}\Omega \right] \\ &\quad - \sum_{j=1}^{m} \mathrm{d} \iiint\limits_{\Omega_j} \left[\int\limits_0^{\mathbf{M}_j} \mathbf{H}_j(\mathbf{M}, T) \cdot \mathrm{d}\mathbf{M} \right] \mathrm{d}\Omega_j.\end{aligned} \tag{38}$$

Using now the identity [8]:

$$\iiint\limits_{\Omega_\infty} \mathbf{H} \cdot \mathbf{B} \, \mathrm{d}\Omega = \sum_{i=1}^{n} I_i \varphi_i, \tag{39}$$

being Ω_∞ the volume of the whole open domain, after some manipulations of the integrals involved into (38), we finally arrive at the following result:

$$\begin{aligned}P_k^{(\mathrm{mag})} &= \frac{\partial A'_{\mathrm{TOT}}}{\partial q_k} = \frac{\mathrm{d}}{\mathrm{d}q_k} \\ &\times \left[\iiint\limits_{\Omega_\infty} \left(\int\limits_0^{\mathbf{H}} \mathbf{B} \cdot \mathrm{d}\mathbf{H} \right) \mathrm{d}\Omega \right]_{\substack{q_r = \text{const., for } r=1..1,\, r \neq k \\ I_i = \text{const., for } i=1..n \\ T_j = \text{const., for } j=1..m \\ T_i = \text{const., for } i=1..n}} \\ &= \frac{\mathrm{d}W'_m}{\mathrm{d}q_k},\end{aligned} \tag{40}$$

having indicated with W'_m the physical quantity:

$$W'_m = \iiint\limits_{\Omega_\infty} \left(\int\limits_0^{\mathbf{H}} \mathbf{B} \cdot \mathrm{d}\mathbf{H} \right) \mathrm{d}\Omega, \tag{41}$$

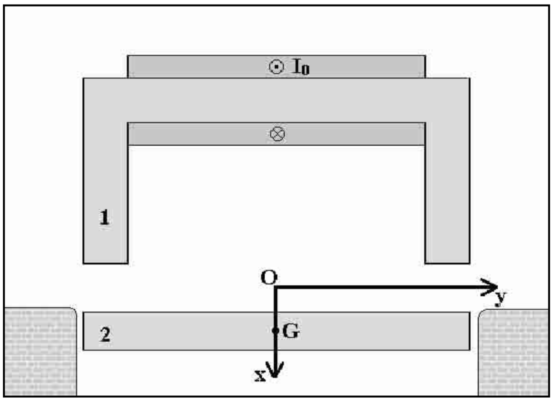

Fig. 2. Test case for the evaluation of magnetic force with the co-energy method.

expressing the so-called *magnetic co-energy*.

Equation (40) proves that the total component $P_k^{(\mathrm{mag})}$ of the magnetic generalized forces along the q_k degree of freedom can be expressed as the derivative of W'_m with respect to q_k. This result is commonly referred to as *co-energy method* for the magnetic force evaluation (see *e.g.* [3]).

It should be stressed again that this result has been obtained under the following constraints:

(i) both ferromagnetic and conducting solids have been considered rigid;
(ii) the integration of **B** in the variable **H** has been carried out keeping constant the currents flowing in the conductors;
(iii) since (**B**, **H**) characteristics depends on temperature, the integration of **B** in the variable **H** has been carried out at constant temperature. This has allowed us considering only the (**B**, **H**) curve corresponding to the starting configuration;
(iv) the ferromagnetic bodies have been considered non-hysteretic, otherwise (37) would not hold, since a fraction of its r.h.s would be dissipated in heat and would not contribute to mechanical work;
(v) as previously recalled, the magnetic constitutive relationships, expressed by (8) and (9), have been assumed such that the integral of **B** in the variable **H** are independent of the way in which **B** goes from zero to its final value (at constant temperature and deformation). This has required that the Jacobian matrix of the functions $\mathbf{B}_j(\mathbf{H}, \bar{\mathbf{E}}, T)$ with respect to the variable **H** be symmetric.

In order to provide an example of application of the co-energy method, let us consider the situation sketched in Figure 2, where an electromagnet is suitably connected to a constant current source I_0. Both the fixed part (element **1** in Fig. 2) and the moving one (element **2** in Fig. 2) of the electromagnet exhibit the same ferromagnetic properties.

If we suppose that the constraints allow the moving part to move only along x-axis, the whole system is characterized by one mechanical degree of freedom q_1.

Indicating with x_G the vertical coordinate of the moving part center of gravity G, we can define $q_1 = x_G$.

Recalling now equation (22), we can express $P_1^{(\text{mag})}$ as

$$P_1^{(\text{mag})} = \sum_{j=1}^{2} \left[\mathbf{F}_j^{(\text{mag})} \cdot \frac{\partial \mathbf{G}_j}{\partial x_G} + \mathbf{M}_{G,j}^{(\text{mag})} \cdot \frac{\partial \varepsilon_j}{\partial x_G} \right] \cdot \quad (42)$$

Since body 1 is fixed and no infinitesimal rotation vectors are allowed by constraints, equation (42) can be simplified into the following:

$$P_1^{(\text{mag})} = \mathbf{F}_2^{(\text{mag})} \cdot \frac{\partial \mathbf{G}_2}{\partial x_G} \cdot \quad (43)$$

Observing now that

$$\frac{\partial \mathbf{G}_2}{\partial x_G} = (0, 1), \quad (44)$$

we can write

$$P_1^{(\text{mag})} = \mathbf{F}_2^{(\text{mag})} \cdot (0, 1) = F_x^{(\text{mag})}, \quad (45)$$

having indicated with $F_x^{(\text{mag})}$ the vertical component of the magnetic force acting upon the moving part of the electromagnet. Finally, $F_x^{(\text{mag})}$ can be evaluated by means of the co-energy method described by equation (40).

4 Magnetic system driven by a permanent magnet

In general, when magnetic systems involving permanent magnets are considered, the methods for computing the resultant force making use of the variations of the magnetic energy (or co-energy) cannot be applied, since hysteretic phenomena are predominant within the magnets themselves. However, there are magnets like modern rare earth magnets (*e.g.* Samarium-cobalt magnets or Neodymium-iron-boron magnets) [12,13], whose demagnetization characteristic is experimentally proved to be linear with excellent approximation (Fig. 3). In such cases, the application of the energy method can be justified by considering that, during the generic virtual displacement $d\mathbf{G}_i$ of ith ferromagnetic body, the operating point O_P of the whole system can move along the demagnetization curve in both directions and in a reversible way. This is due to the fact that demagnetization and recoil curves are coincident for this kind of materials whereas the same does not hold for materials exhibiting nonlinear demagnetization curves. In such cases, the operating point can also move on different (linear) recoil curves, which are dependent on the previous magnetization states. Therefore, in order to apply the energy method for a given operating point, the actual recoil curve should be a priori known and it should be verified that the operating point itself lies only on this recoil curve during the virtual displacement.

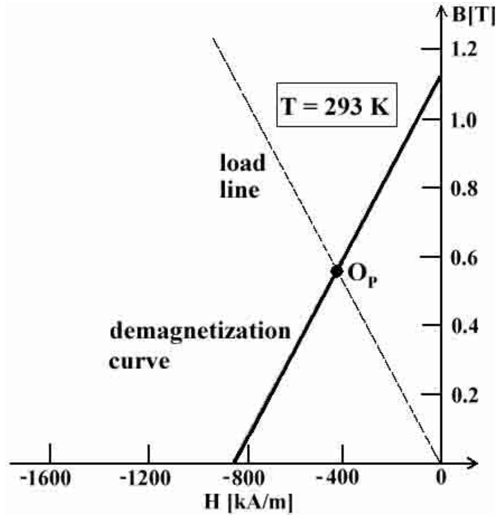

Fig. 3. Demagnetization characteristics and generic operation point of a fully dense $Nd_2Fe_{14}B$ permanent magnet.

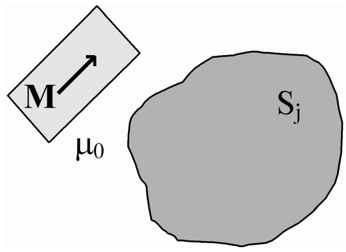

Fig. 4. Geometry of the model problem driven by the field of a permanent magnet.

The assumption of linear model makes the hypothesis of operating point moving only on the demagnetization curve consistent with physical reality.

In practice, the way in which the magnet has been magnetized is not taken into account and every hysteretic behavior in the second quadrant of the (B,H) plane is neglected. Once the model has been defined, it is possible to formulate a mathematical theory, like that developed in the previous section, in order to show the validity of MEM.

To this purpose, let us imagine to remove the n conducting solids C_1, C_2, ..., C_n from the system of Figure 1, adding simultaneously a permanent magnet acting as a field source (Fig. 4).

In order to apply the previous approach to this new configuration, it is necessary to evaluate the variation of the total Helmholtz free energy $A_{\text{TOT, PM}}$ in the magnet. This expression can be obtained by a straightforward application of the arguments developed in the previous section.

As is well-known, a linear permanent magnet is characterized by the following relationship:

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} + \mathbf{B}_r, \quad (46)$$

and its magnetization $\mathbf{M}$ can be generally expressed as

$$\mathbf{M} = \mathbf{B} - \mu_0 \mathbf{H}. \quad (47)$$

Inserting (46) into (47), one has

$$\mathbf{M} = (\mu_r - 1)\mu_0\,\mathbf{H} + \mathbf{B}_r. \tag{48}$$

The following two situations can occur:

1. $\mu_r = 1$ (ideal permanent magnet).
2. $\mu_r \neq 1$.

Let us demonstrate that in both cases the same expression for $\mathrm{d}A_{\mathrm{TOT,\,PM}}$ is obtained.

1. Equation (48) dictates that $\mathbf{M}$ is constant and equal to $\mathbf{B}_r$. Therefore the variation of the total Helmholtz free energy $\mathrm{d}A_{\mathrm{TOT,\,PM}}$ on the volume Ω_{PM} occupied by the permanent magnet can be easily computed integrating over Ω_{PM} equation (6), where the term involving the integral of $\mathbf{H}$ in the variable $\mathbf{M}$ vanishes:

$$\begin{aligned}\mathrm{d}A_{\mathrm{TOT,\,PM}} &= \mathrm{d}\iiint\limits_{\Omega_{\mathrm{PM}}} \frac{1}{2}\mu_0 H^2 \mathrm{d}\Omega - S^{(\mathrm{PM})}\mathrm{d}T^{(\mathrm{PM})}\\ &= \mathrm{d}\iiint\limits_{\Omega_{\mathrm{PM}}}\left[\int_{\mathbf{B}_0}^{\mathbf{B}} \mathbf{H}(\mathbf{B},T)\cdot\mathrm{d}\mathbf{B}\right]\mathrm{d}\Omega - S^{(\mathrm{PM})}\mathrm{d}T^{(\mathrm{PM})}\end{aligned} \tag{49}$$

where $\mathbf{B}_0$ is an arbitrary starting point on the $\mathbf{H} = \mathbf{H}(\mathbf{B})$ curve and $S^{(\mathrm{PM})}$ and $T^{(\mathrm{PM})}$ are respectively the entropy and the absolute temperature of the magnet.

It should be underlined that $\mathbf{B}_0$ is arbitrary since we are only interested to the variation of the total free energy of the magnet.

2. In this case the variation of $A_{\mathrm{TOT,\,PM}}$ becomes

$$\begin{aligned}\mathrm{d}A_{\mathrm{TOT,\,PM}} &= \mathrm{d}\left[\iiint\limits_{\Omega_{\mathrm{PM}}} \frac{1}{2}\mu_0 H^2\mathrm{d}\Omega\right]\\ &\quad + \mathrm{d}\iiint\limits_{\Omega_{\mathrm{PM}}}\left[\int_0^{\mathbf{M}}\mathbf{H}(\mathbf{M},T)\cdot\mathrm{d}\mathbf{M}\right]\mathrm{d}\Omega - S^{(\mathrm{PM})}\mathrm{d}T^{(\mathrm{PM})}\\ &= \mathrm{d}\iiint\limits_{\Omega_{\mathrm{PM}}}\left[\frac{1}{2}\mu_0 H^2 + \frac{1}{2}(\mu_r-1)\mu_0 H^2 - \frac{1}{2}\frac{B_r^2}{\mu_r(\mu_0-1)}\right]\\ &\qquad \times\mathrm{d}\Omega - S^{(\mathrm{PM})}\mathrm{d}T^{(\mathrm{PM})}\\ &= \mathrm{d}\iiint\limits_{\Omega_{\mathrm{PM}}}\left[\frac{1}{2}\mu_0 H^2 + \frac{1}{2}(\mu_r-1)\mu_0 H^2\right]\mathrm{d}\Omega - S^{(\mathrm{PM})}\mathrm{d}T^{(\mathrm{PM})}\\ &= \mathrm{d}\iiint\limits_{\Omega_{\mathrm{PM}}}\left[\int_{\mathbf{B}_0}^{\mathbf{B}}\mathbf{H}(\mathbf{B},T)\cdot\mathrm{d}\mathbf{B}\right]\mathrm{d}\Omega - S^{(\mathrm{PM})}\mathrm{d}T^{(\mathrm{PM})}.\end{aligned} \tag{50}$$

Now, we can write an expression analogous to (31) for the system of Figure 4:

$$\begin{aligned}-\sum_{k=1}^{l} P_k^{(\mathrm{mag})}\delta q_k &= \mathrm{d}W_{\mathrm{f}} + \sum_{j=1}^{m}\mathrm{d}A_j^{(\mathrm{S})}\\ &\quad + \sum_{j=1}^{m} S_j^{(\mathrm{S})}\mathrm{d}T_j^{(\mathrm{S})} + \mathrm{d}A_{\mathrm{PM}} + S_j^{(\mathrm{PM})}\mathrm{d}T_j^{(\mathrm{PM})}\\ &= \mathrm{d}A_{\mathrm{TOT}} + \sum_{j=1}^{m} S_j^{(\mathrm{S})}\mathrm{d}T_j^{(\mathrm{S})} + S_j^{(\mathrm{PM})}\mathrm{d}T_j^{(\mathrm{PM})},\end{aligned} \tag{51}$$

and we can express $P_k^{(\mathrm{mag})}$ as

$$\begin{aligned}P_k^{(\mathrm{mag})} &= -\frac{\partial A_t}{\partial q_k} = -\frac{\mathrm{d}}{\mathrm{d}q_k}\Bigg[\iiint\limits_{\Omega_\infty - \Omega_{\mathrm{PM}}}\left(\int_0^{\mathbf{B}}\mathbf{H}\cdot\mathrm{d}\mathbf{B}\right)\mathrm{d}\Omega\\ &\qquad + \iiint\limits_{\Omega_{\mathrm{PM}}}\left[\int_{\mathbf{B}_0}^{\mathbf{B}}\mathbf{H}(\mathbf{B},T)\cdot\mathrm{d}\mathbf{B}\right]\\ &\qquad \times\mathrm{d}\Omega\Bigg]_{\substack{q_r=\text{const., for } r=1..1,\, r\neq k\\ T_j=\text{const., for } j=1..m\\ T_{\mathrm{PM}}=\text{const.}}}\\ &= -\frac{\mathrm{d}W_m}{\mathrm{d}q_k},\end{aligned} \tag{52}$$

having indicated with W_m the physical quantity

$$\begin{aligned}W_m &= \iiint\limits_{\Omega_\infty - \Omega_{(\mathrm{PM})}}\left(\int_0^{\mathbf{B}}\mathbf{H}(\mathbf{B},T)\cdot\mathrm{d}\mathbf{B}\right)\mathrm{d}\Omega\\ &\quad + \iiint\limits_{\Omega_{(\mathrm{PM})}}\left(\int_{\mathbf{B}_0}^{\mathbf{B}}\mathbf{H}(\mathbf{B},T)\cdot\mathrm{d}\mathbf{B}\right)\mathrm{d}\Omega,\end{aligned} \tag{53}$$

expressing the so-called *magnetic energy* for a system involving a permanent magnet. In strict analogy with the situation examined in the previous section, equation (53) proves that the total component $P_k^{(\mathrm{mag})}$ of the magnetic generalized forces along the q_k degree of freedom can be expressed as the derivative of W_m with respect to q_k. This procedure is commonly referred to as *energy method* for the magnetic force evaluation (see *e.g.* [3]).

5 Conclusions

A proof of the validity of the so-called energy and coenergy methods for the evaluation of the total force acting upon a magnetized body in a magnetostatic field created by current sources or permanent magnets has been provided in this paper. The demonstration, carried out in the general framework of a comprehensive theory including not only magnetic and mechanical energy transfers, but also thermodynamic energy balances, has pointed out the limitations of the methods and the real conditions under which they can be applied when dealing with electromechanical devices made by non-linear and non-hysteretic ferromagnetic materials.

Appendix

In this appendix we briefly recall some general definitions of rational mechanics, which can be useful for a better understanding of some mathematical notations adopted throughout the paper. All the concepts here mentioned are treated in details in [14,15].

DEF. 1: A *particle* is an entity whose motion is completely specified by a continuous point. It is common usage in continuum mechanics to indicate the material points of a body as particles. Therefore, the expression "the particle X" is frequently adopted in lieu of "the point at the position $\mathbf{X}$ at $t = 0$".

DEF. 2: Suppose that the positions of the n particles of a system are all specified relative to a given set of Cartesian axes, so that (x_i, y_i, z_i) are the coordinates of the ith particle. We can define a *configuration space* for the system by representing the configuration of the n particles, at any time t, by the point in $3n$-dimensional Euclidean space having Cartesian coordinates $(\mathbf{X}^1, ..., \mathbf{X}^{3n}) = (x_1, y_1, z_1, ..., x_n, y_n, z_n)$.

DEF. 3: A *constraint* is defined to be a relation between $\mathbf{X}^i$, $\mathrm{d}\mathbf{X}^i/\mathrm{d}t$ and the time t, not involving the accelerations $\mathrm{d}^2\mathbf{X}^i/\mathrm{d}t^2$.

DEF. 4: A constraint consisting of one or more equations of the kind:

$$\Phi_\mu(\mathbf{X}^i, \dot{\mathbf{X}}^i, t) = 0, \quad \mu = 1, ..., k, \tag{A.1}$$

where the functions Φ_μ are linear in $\dot{\mathbf{X}}^i$ and in general have continuous partial derivatives of the first order, is said to be *holonomic* if these equations can be integrated to give

$$f_\mu(\mathbf{X}^i, t) = 0, \quad \mu = 1, ..., k, \tag{A.2}$$

where the differentiable functions f_μ do not involve the velocities $\dot{\mathbf{X}}^i$. The simplest holonomic constraint is that which requires a set of particles to form a *rigid* system.

DEF. 5: Suppose that a system of n particles may be subjected to constraints of arbitrary character. If any of these constraints are holonomic, then some, or all, of the corresponding equations (A.2) can be used to eliminate some of the coordinates $\mathbf{X}^i$. Therefore, the remaining coordinates will still serve to specify the configuration of the system. Any set of independent parameters $q_1, ..., q_l$ whose values at any time, together with the value of t, completely specify the configuration of the system are known as *generalized coordinates* for the system.

We wish to remember Professor Scipione Bobbio, who was for us an enlightening guide in the study of electrodynamics of materials and who conveyed us his passion for electromagnetism.

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